Fermi National Accelerator Laboratory

FERMILAB-PUB-92/192-T

FOUR LOOP PERTURBATIVE CALCULATIONS of $\sigma_{tot}(e^+e^- \to hadrons)$, $\Gamma(\tau^- \to \nu_\tau + hadrons)$ and QED β-function

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Abstract

We present the description of the methods and the calculation of the ratio: $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ at the four-loop level of perturbative QCD. The calculation of $O(\alpha_s^3)$ QCD contribution to the $\Gamma(\tau^- \to \nu_\tau + hadrons)$ and the four-loop QED renormalization group β function are also described. An analysis of the scheme-scale dependence of the results within 't Hooft's minimal subtruction prescription is done. We apply the three most effective approaches for resolving the scheme-scale ambiguity and we fix the scale for which all of the criteria tested are satisfied. The variant of the MS-type scheme is suggested. The theoretical uncertainty of the strong interaction effects in $R(s)$ is estimated at 4%.

Shortened version submitted to the 26th International Conference on High Energy Physics, Dallas, USA, August, 1992.

Contents

1 Introduction

The most impressive confirmation of the standard theory was obtained from experiments at e^+e^- colliders within last decade. The precise experimental tests of the theory of strong interactions (QCD) [1] confirms the fundamental postulates of the theory.

Much information has been extracted from measurements of the characteristics of the process of e^+e^- annihilation into hadrons [2]. The most convenient characteristic of the process, the ratio: $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ was studied within the available region of energy ($\sqrt{s} < 94GeV$). First this process provides a fundamental QCD test, providing evidence for the existence of color, which has been introduced in [3]. Moreover, this process clearly proves existence of c ($\sqrt{s} = 3GeV$) and b ($\sqrt{s} = 10GeV$) quarks and the measurements of $R(s)$ at high energies gives a restriction on the existence of new quarks in the corresponding regions [4]. The e^+e^- annihilation is also the basic channel for studying heavy quark production. The comparison of the theoretical calculations and the experimental results allow one to extract the basic parameters of the theory, such as the strong coupling constant $\alpha_s(Q^2)$ and QCD scale parameter Λ_{QCD} [2, 5]. Note also, that the ratio $R(s)$ plays a key role in the QCD sum rule method, which is widely used in resonance physics and also for estimation of important QCD parameters, such as vacuum condensates, quark masses etc. [6].

The fundamental physical property of QCD - asymptotic freedom [1] enables one to study with confidence the high-energy behavior of the processes involving large momentum transfer. In particular, the ratio $R(s)$ can be calculated theoretically in the deep euclidean region within the framework of perturbative QCD, by using the renormalization group (RG) formalism [7].

The result of the naive parton approach looks like: $R_0 = 3\sum_{i=1}^{n} Q_i^2$, where the sum runs over the excited quark charge and the factor 3 stands for the color degree of freedom of each quark flavors. Note, that this result does not deal with QCD. The effects of strong interactions come as perturbations and should be taken into account by a multiplicative factor modifying the parton result as: $R(s) = R_0*(1+\sum_{n=1}^{\infty} f_n(\alpha_s/\pi)^n)$. The calculation of the perturbative coefficients f_n is an important problem from a viewpoint of estimation of the strong interaction effects on R. On the other hand it raises an important question on the behavior of the corresponding perturbation series and as a result the problem on the applicability of perturbative QCD to calculations of physical quantities. The dimensional regularization method [8] and the corresponding renormalization procedure [9] (see also [7]) in combination with the recent progress in calculational algorithms for relevant types of Feynman diagrams [10], the infrared rearrangement technique [11, 12, 13] and their computer realization [14, 15, 16, 17] (the algebraic programming systems used are REDUCE

[18], SCHOONSCHIP [19] and FORM [20]) allow one to perform the RG-calculations to the four-loop level.

The leading QCD correction to R was computed a long time ago in the zero-quark-mass limit [1]. The $O(\alpha_s)$ correction for massive quarks has also been calculated [21].

Using the methods and algorithms mentioned above, the three-loop correction to R was computed analytically in ref. [22, 23]) and numerically in ref. [24]. As was analyzed in [22], the numerical computation of each Feynman diagram was done with high accuracy, however due to cancellations between different diagrams within their gauge invariant sets, the error in the final result reaches several percents. That is one of the main points of importance of the exact analytical calculations.

Four years ago, the results of calculations of next-next to leading QCD corrections to $R(s)$ was given in ref. [25]. The numerical value of the four-loop correction was very large. This cast doubt on the feasibility of obtaining reliable estimations for the QCD corrections to $R(s)$ via perturbation theory. (The same effect was observed in perturbative calculations of the coefficient functions of condensates in QCD sum rules [26].) Inclusion of the $O(\alpha_s^3)$ correction changed the value of α_s by about 10%. The corresponding value of Λ_{MS} decreases drastically (by a factor of 2). However, further consideration shows that the result of ref. [25] for the four-loop correction is not correct. (The causes are discussed below.) Thus, in ref. [27] we presented new results of an independent reevaluation of the four-loop correction to $R(s)$. Our result is much smaller and has the opposite sign compared to the incorrect result [25]. Later, our result [27] was confirmed in ref. [28]. [1]

In this paper we describe the technology of our calculations of $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ at the four-loop level of perturbative QCD. (The results of these calculations we have briefly reported in our recent publications [27].) Using the method of ref. [29, 30] and the results of our calculations, we obtained the $O(\alpha_s^3)$ perturbative QCD corrections to the ratio: $R_\tau = \Gamma(\tau^- \to \nu_\tau + hadrons)/\Gamma(\tau^- \to \nu_\tau e^- \overline{\nu}_e)$. (This result was briefly reported in ref. [31].) As an intermediate result, the four-loop QED β - function was computed. The scheme-scale dependence of the results and the obtained perturbative series are also analyzed.

The plan of the present paper is as follows. In sect.2 the outline of the calculational methods and the corresponding formulae is given. In sect.3 key details and the results of the calculation of $R(s)$ at the four-loop level are presented. Sect.4 is dedicated to the description of calculations of the $O(\alpha_s^3)$ order corrections to the ratio R_τ. The calculation of the four-loop renormalization group functions in QED is described in sect.5. Sect.6 contains the detailed analysis of the obtained perturbative series

[1] Unfortunately in ref. [28] the corresponding citation on ref. [27] is absent.

and the scheme-scale dependence of the results is discussed. The accuracy of the obtained results for the physical observables is estimated. The paper is completed with concluding remarks.

2 General Relations and Calculational Methods Used

2.1 Notation

We consider the standard QCD [1], Lagrangian which is:

$$L(x) = -1/4(G^a_{\mu\nu})^2 - 1/2(\partial_\mu A^a_\mu)^2 + \sum_f \overline{q}_f(i\hat{\partial} - m_f)q_f + g\sum_f \overline{q}_f T^a \hat{A}^a q_f + \partial_\mu \overline{c}^a(\partial_\mu \delta^{ac} + g f^{abc} A^b_\mu)c^c \qquad (1)$$

$G^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu$, $\hat{\partial} = \gamma^\mu \partial_\mu$, $\hat{A}^a = \gamma^\mu A^a_\mu$, where c^a are the Faddeev-Popov ghosts, A^a and q_f are gluon and quark fields and m_f are the quark masses. The index f enumerates the quark flavors, total number of which is N_f. The generators T^a of the $SU(N)_c$ gauge group, the structure constants f^{abc} and d_{abc} obey the properties:

$$[T^a, T^b] = i f^{abc}, \;\; f^{acd} f^{bcd} = C_a \delta^{ab}, \;\; tr T^a T^b = T\delta^{ab}, \;\; T^a T^a = C_F * \hat{1}, \;\; tr\hat{1} = N_F, \;\; \delta^{aa} = N_A \qquad (2)$$

For the fundamental representation of $SU(3)_c$:

$$C_A = 3, \;\; C_F = 4/3, \;\; T = 1/2, \;\; N_F = 3, \;\; N_A = 8, \;\; d_{abc}d_{abc} = 40/3. \qquad (3)$$

Throughout this work we use standard QCD Feynman rules [1], dimensional regularization [8] and the $\overline{MS}$ scheme [32] for ultraviolet renormalization. The three-loop QCD β function has the form [33]:

$$\beta(\alpha_s) = -\beta_0 \alpha_s/4\pi - \beta_1(\alpha_s/4\pi)^2 - \beta_2(\alpha_s/4\pi)^3 + O(\alpha_s^4), \qquad (4)$$

where $\beta_0 = 11/3C_A - 4/3TN_f$;
$\beta_1 = 34/3C_A^2 - 20/3C_A T N_f - 4C_F T N_f$;
$\beta_2 = 2857/54C_A^3 - 1415/27C_A^2 T N_f + 158/27C_A T^2 N_f^2 - 205/9C_A C_F T N_f + 44/9C_F T^2 N_f^2 + 2C_F^2 T N_f$.
The standard parametrization for the running coupling to three-loops looks like:

$$\frac{\alpha_s(s)}{4\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1 ln L}{\beta_0^3 L^2} + \frac{1}{\beta_0^5 L^3}(\beta_1^2 ln^2 L - \beta_1^2 ln L + \beta_2\beta_0 - \beta_1^2) + O(L^{-4}) \qquad (5)$$

where $L = ln(s/\Lambda^2)$. Parametrization (5) has the same form and the QCD β-function coefficients are the same, at least within MS-type schemes. The scale parameter Λ depends on the renormalization scheme chosen.

2.2 Hadronic vacuum polarization function

Consider the hadronic vacuum polarization function defined as:

$$\Pi_{\mu\nu}(Q) = i \int e^{iqx} < T j_\mu(x) j_\nu(0) >_0 d^4x = (q_\mu q_\nu - g_{\mu\nu} q^2)\Pi(Q^2) \qquad (6)$$

where $j_\mu(x) = \overline{\psi}\gamma_\mu\psi$ and $Q^2 = -q^2$ is the euclidean momentum squared.

The well known dispersion relation allows one to connect $R(s)$ to the hadronic vacuum polarization function:

$$D(Q^2) = -\frac{3}{4}\frac{\partial}{\partial ln Q^2}\Pi(Q^2) = Q^2 \int_0^\infty \frac{R(s)}{(s+Q^2)^2} ds \qquad (7)$$

(Derivative here avoids inconvenient extra subtraction in the r.h.s.) where the function $D(Q^2)$, which is introduced for convenience, satisfies the RG-equation [7]:

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} - \gamma_m(\alpha_s)\frac{\partial}{\partial ln m}\right) D(Q^2/\mu^2, m, \alpha_s) = 0 \qquad (8)$$

where μ is 't Hooft's unit of mass [8] and γ_m is the anomalous dimension function of the quark mass, known at present at the three-loop level [34].

Acording to the operator product expansion technique [35] one can separate pure perturbative and nonperturbative contributions to the function $\Pi(Q^2)$. Indeed, as shown in ref. [6], this function can be represented as:

$$\Pi(Q^2) = (Perturbation\ Theory) + \sum_{n\geq 2} \frac{C_n(Q) < O_n >_0}{Q^{2n}} + (instanton\ contrib.) \qquad (9)$$

where $< O_n >_0$ denotes vacuum condensates and $C_n(Q)$ it's coefficient functions. The last term in the above equation describes the instanton contributions, which were evaluated in ref. [36]. High order perturbative corrections to the coefficient functions of the dimension 4 power terms (gluon and quark condensates) in eq.(9) has been calculated in ref.[26] and for the dimension 6 power terms in ref. [37]. Note, that we will consider the region of very high energies where, in fact, only perturbation theory

contributions survive in $R(s)$. The first term in eq.(9) can be expanded in powers of m^2/Q^2 at high energies (at $m^2/Q^2 \to 0$):

$$D(Q^2/\mu^2, m, \alpha_s) = \sum_{i\geq 0} D_i\left(\frac{Q^2}{\mu^2}, \alpha_s\right)\left(\frac{m^2}{Q^2}\right)^i \tag{10}$$

It is easy to see that terms of the type m^2/Q^2 have the perturbative nature. The coefficient function of the next to leading term in eq.(10) has been calculated in ref. [38] to the three-loop level. Unfortunately, the result of [38] contains an error, which fortunately is numerically small. The correct analytical result for this three-loop coefficient function has been presented in [39]. Note, that apparently there exists region of values of Q^2, where power corrections (especially of the type m^{2n}/Q^{2n} for the heavy quarks) becomes important.

At the first step, we neglect all contributions to $\Pi(Q^2)$, except perturbation theory and take all quark masses to be zero. So, we will deal with $D_0(Q^2/\mu^2, \alpha_s)$.

2.3 Renormalization relations

The solution of eq.(8) is (here and below we set $D_0 \equiv D$):

$$D(Q^2/\mu^2, \alpha_s) = D(\overline{\alpha}_s(Q^2)) = \sum_{i\geq 0} R_i(\overline{\alpha}_s/\pi)^i \tag{11}$$

where the $\overline{\alpha}_s(Q^2)$ is the running coupling.

It is known that the vacuum polarization function is renormalized additively:

$$\Pi(Q^2/\mu^2, \alpha_s) = \Pi^B(Q^2/\mu^2, \alpha_s^B) + Z \equiv finite \tag{12}$$

The bare coupling α_s^B is related to the renormalized one by the relation:

$$\alpha_s^B = \alpha_s\left[1 - \frac{\alpha_s}{4\pi}\frac{\beta_0}{\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon}\right) + O(\alpha_s^3)\right] \tag{13}$$

Here $\varepsilon = (4 - D)/2$ and the D is the space-time dimension. Recall, that within the minimal subtraction prescription [9] the renormalization constant Z can be expressed as the following double sum [7]:

$$Z = \sum_{-l\leq k<0}\left(\frac{\alpha_s}{4\pi}\right)^{l-1} Z_{lk}\varepsilon^k \tag{14}$$

where Z_{lk} are numbers. Furthermore, for the bare vacuum polarization function we have the expansion in perturbation series:

$$\Pi^B\left(\frac{Q^2}{\mu^2},\alpha_s^B\right)=\sum_{-l\le k,l>0}\left(\frac{\alpha_s^B}{4\pi}\right)^{l-1}\left(\frac{\mu^2}{Q^2}\right)^{le}\Pi_{lk}\epsilon^k \tag{15}$$

where, the first index denotes the number of loops of the corresponding Feynman diagrams.

Substituting eq.(15) into the definition (7) and recalling the relations (11),(13), we get:

$$D(\overline{\alpha}_s(Q^2))=\frac{3}{4}\left(\Pi_{1,-1}+\frac{\overline{\alpha}_s}{4\pi}\left[2\Pi_{2,-2}\frac{1}{\varepsilon}+2\Pi_{2,-1}\right]\right.$$

$$+\left(\frac{\overline{\alpha}_s}{4\pi}\right)^2\left[\frac{1}{\varepsilon^2}\left(3\Pi_{3,-3}-2\beta_0\Pi_{2,-2}\right)+\frac{1}{\varepsilon}\left(3\Pi_{3,-2}-2\beta_0\Pi_{2,-1}\right)+\left(3\Pi_{3,-1}-2\beta_0\Pi_{2,0}\right)\right]$$

$$\begin{aligned}&+\left(\frac{\overline{\alpha}_s}{4\pi}\right)^3\left[\frac{1}{\varepsilon^3}\left(4\Pi_{4,-4}-6\beta_0\Pi_{3,-3}+2\beta_0^2\Pi_{2,-2}\right)\right.\\&+\frac{1}{\varepsilon^2}\left(4\Pi_{4,-3}-6\beta_0\Pi_{3,-2}-\beta_1\Pi_{2,-2}+2\beta_0^2\Pi_{2,-1}\right)\\&+\frac{1}{\varepsilon}\left(4\Pi_{4,-2}-6\beta_0\Pi_{3,-1}-\beta_1\Pi_{2,-1}+2\beta_0^2\Pi_{2,0}\right)\\&\left.\left.+\left(4\Pi_{4,-1}-6\beta_0\Pi_{3,0}-\beta_1\Pi_{2,0}+2\beta_0^2\Pi_{2,1}\right)\right]\right)\\&+O(\alpha_s^4)\equiv finite\end{aligned} \tag{16}$$

Furthermore, substituting (13)-(15) into (12), we obtain:

$$\begin{aligned}&div\Pi(\overline{\alpha}_s(Q^2))=\\&\frac{1}{\varepsilon}(\Pi_{1,-1}+Z_{1,-1})+\frac{\overline{\alpha}_s}{4\pi}\left[\frac{1}{\varepsilon^2}(\Pi_{2,-2}+Z_{2,-2})+\frac{1}{\varepsilon}(\Pi_{2,-1}+Z_{2,-1})\right]\\&+\left(\frac{\overline{\alpha}_s}{4\pi}\right)^2\left[\frac{1}{\varepsilon^3}(\Pi_{3,-3}-\beta_0\Pi_{2,-2}+Z_{2,-2})\right.\\&+\frac{1}{\varepsilon^2}(\Pi_{3,-2}-\beta_0\Pi_{2,-1}+Z_{3,-2})+\frac{1}{\varepsilon}(\Pi_{3,-1}-\beta_0\Pi_{2,0}+Z_{3,-1})\left.\right]\\&+\left(\frac{\overline{\alpha}_s}{4\pi}\right)^3\left[\frac{1}{\varepsilon^4}(\Pi_{4,-4}-2\beta_0\Pi_{3,-3}+\beta_0^2\Pi_{2,-2})\right.\\&\left.+\frac{1}{\varepsilon^3}(\Pi_{4,-3}-2\beta_0\Pi_{3,-2}+\beta_0^2\Pi_{2,-1}-\beta_1\Pi_{2,-2}/2+Z_{4,-3})\right.\end{aligned} \tag{17}$$

$$+\frac{1}{\varepsilon^2}(\Pi_{4,-2} - 2\beta_0\Pi_{3,-1} + \beta_0^2\Pi_{2,0} - \beta_1\Pi_{2,-1}/2 + Z_{4,-2})$$
$$+\frac{1}{\varepsilon}(\Pi_{4,-1} - 2\beta_0\Pi_{3,0} + \beta_0^2\Pi_{2,1} - \beta_1\Pi_{2,0}/2 + Z_{4,-1})\Big] + O(\alpha_s^4)$$

2.4 Evaluation of the renormalization constants

We now turn to a brief discussion of the evaluation of renormalization constants within 't Hooft's scheme [9], using Vladimirov's method [11] and the so-called Infrared rearrangement procedure [12, 13].

To calculate the renormalizaton constant Z_Γ to the one-particle-irreducible Green's function Γ it is convenient to use the following representation [11]:

$$Z_\Gamma = 1 - \mathcal{K}R'\Gamma \tag{18}$$

where $\mathcal{K}$ picks out all singular terms from the Laurent series in ε:

$$\mathcal{K}\sum_i c_i\varepsilon^i = \sum_{i<0} c_i\varepsilon^i$$

and R' is defined by the recursive relation:

$$R'G = G - \sum \mathcal{K}R'G_1 \ldots \mathcal{K}R'G_n \times G_{/(G_1+\ldots+G_n)} \tag{19}$$

where a sum is running over all sets of one-particle-irreducible divergent subgraphs of G and $G_{/(G_1+\ldots+G_n)}$ is the diagram G with the subgraphs $G_1 + \ldots + G_n$ shrinked to a point. Actually, R' is the ordinary Bogoliubov-Parasiuk R-operation [1, 7] without the last subtraction. So, R' subtracts all "internal" divergences only and is connected to the ordinary R-operation as:

$$R = (1 - \mathcal{K})R'.$$

Thus, to calculate the renormalization constants Z_{ij} one should write the diagram representation of Π and apply $\mathcal{K}R'$ to the corresponding graphs or, in other words, one should evaluate the counterterm for each graph.

As we shall see below, the benefit of using the relation (18) is based on the following fact, that the $\mathcal{K}R'$ for each diagram is a polynomial in dimensional parameters. This fundamental property of the 't Hooft's minimal subtraction prescription is the basic idea of the various versions of the infrared rearrangement (IR) technique [11, 12, 13], the essence of which is as follows. Using that fact, that $\mathcal{K}R'$ is a polynomial in masses and external momenta of the diagram, one can remove dependence on the external momenta by differentiating (usually twice is sufficient) with respect to the external momentum and then external momentum is set to zero. However, in this case infrared

divergences appear. In order to prevent this, one should introduce a new fictitious external momentum as an infrared regulator flowing along some line of the diagram.

The main result of application of the IR-technique [11, 12, 13] is that the problem of calculating the counterterm of an arbitrary l-loop diagram with an arbitrary number of masses and external momenta within the MS-prescription can be reduced, through infrared rearrangement, to the problem of calculating some $l-1$ -loop massless integrals to $O(\varepsilon^0)$ with only one external momentum. In the next section the full calculational procedure will be demonstrated for a typical four-loop diagram, contributing to Z.

3 $R(s) = \sigma_{tot}(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$ at the 4-loop Level of Perturbative QCD

3.1 $R(s)$ via renormalization constants

The vacuum polarization function $\Pi(Q^2)$ has a cut along the negative Q^2 axis (for the massless case). The ratio $R(s)$ is the discontinuity of the $\Pi(Q^2)$ across this cut [1]:

$$R(s) = \frac{1}{2\pi i}(\Pi(-s+i\varepsilon) - \Pi(-s-i\varepsilon)) \tag{20}$$

Integrating (7) and substituting the RG-improved perturbation theory expansion (11), after analytical continuation of the results into the physical region (see: [40, 41, 42]), we obtain to $O(\alpha_s^3)$:

$$R(s) = R_0 + \frac{\overline{\alpha}_s(Q^2)}{\pi}R_1 + (\frac{\overline{\alpha}_s(Q^2)}{\pi})^2 R_2 + (\frac{\overline{\alpha}_s(Q^2)}{\pi})^3(R_3 - R_1\pi^2\beta_0^2/3) \tag{21}$$

The last term in this equation is scheme independent and is a result of the procedure of analytical continuation of the results of perturbative calculation of $d\Pi(Q^2)/dQ^2$ to the physical region. (for details see: [40, 41, 42]). Note, that R_i in the above equation is the perturbative coefficients of the $D(Q^2)$ function (see eq.(11)).

We are going now to obtain some of the key relations for the renormalization constants. Here we follow the procedure, which has been described in ref. [43] at the three-loop level. Let's analyze the expression (16). It is easy to see, that the condition $divD(Q^2) = 0$, automatically gives $\Pi(2,-2) = 0$. Then, analyzing the orders $O(\alpha_s^2)$ and $O(\alpha_s^3)$, one can obtain $\Pi(3,-3) = 0$ and $\Pi(4,-4) = 0$. These relations with the condition $div\Pi(Q^2) = 0$ gives: $Z_{2,-2} = 0$; $Z_{3,-3} = 0$ and $Z_{4,-4} = 0$. The above relations mean that the leading poles should cancel in the sum of all relevant diagrams at a given order of α_s. Actually, as the explicit calculations show, the leading poles cancel within each gauge invariant set of diagrams. This fact serves as one of the

useful checks of the calculations. From the condition of cancellation of the poles in (17) we get the following two sets of relations:

$$\begin{aligned} 3Z_{3,-2} + \beta_0 Z_{2,-1} &= 0 \\ 2Z_{4,-3} + \beta_0 Z_{3,-2} &= 0 \\ 2Z_{4,-2} + \beta_0 Z_{3,-1} + \beta_1 Z_{2,-1}/2 &= 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} \Pi_{1,-1} &= -Z_{1,-1} \\ \Pi_{2,-1} &= -Z_{2,-1} \\ \Pi_{3,-2} &= -Z_{3,-2} - \beta_0 Z_{2,-1} \\ \Pi_{3,-1} &= -Z_{3,-1} + \beta_0 \Pi_{2,0} \\ \Pi_{4,-1} &= -Z_{4,-1} + 2\beta_0 \Pi_{3,0} + \beta_1 \Pi_{2,0}/2 - \beta_0^2 \Pi_{2,1} \\ \Pi_{4,-2} &= -Z_{4,-2} - 2\beta_0 Z_{3,-1} - \beta_1 Z_{2,-1}/2 + \beta_0^2 \Pi_{2,0} \\ \Pi_{4,-3} &= -Z_{4,-3} - 2\beta_0 Z_{3,-2} - \beta_0^2 Z_{2,-1} \end{aligned} \tag{23}$$

The first relation in (22) and first four relations in (23) coincide with the analogous relations obtained in the three-loop calculations previously [22, 43]. The set (22) also serves as a useful test of the calculation.

Substituting (23) in (16) and recalling (21) we obtain our main expression for $R(s)$:

$$\begin{aligned} R(s) = & \\ & -\frac{3}{4}\left[Z_{1,-1} + \frac{\overline{\alpha}_s}{4\pi}(2Z_{2,-1}) + \left(\frac{\overline{\alpha}_s}{4\pi}\right)^2 (3Z_{3,-1} - \beta_0 \Pi_{2,0}) \right. \\ & \left. + \left(\frac{\overline{\alpha}_s}{4\pi}\right)^3 (4Z_{4,-1} - 2\beta_0 \Pi_{3,0} - \beta_1 \Pi_{2,0} + 2\beta_0^2 \Pi_{2,1} - 2Z_{2,-1}\pi^2 \beta_0^2/3 + O(\alpha_s^4)) \right] \end{aligned} \tag{24}$$

The expression obtained for $R(s)$ shows that in order to calculate the l-loop contribution to R, one should calculate the l-loop counterterm Z to the bare quantity Π^B, and the $l-1$ -loop approximation to Π^B. So, in our case we deal with calculations of massless one-, two- and three-loop Feynman diagrams up to their finite part and the divergent parts of the four-loop diagrams.

The algorithm [10] and the computer programs [14, 15] allow one to calculate propagator type Feynman diagrams to three-loop level only, up to their finite parts. So,

it is impossible to do a straightforward calculation of the divergent parts of the necessary four-loop diagrams, without extension of the above algorithms and programs to four-loop order.[2] In this case more refined methods should be used. In the previous section we have briefly mentioned the so called infrared rearrangement procedure [11, 12, 13]. Through this procedure the problem of calculation of the counterterm to the arbitrary l -loop diagram with an arbitrary number of masses and external momenta within the MS prescription [9] can be reduced, to the problem of calculating some $l-1$ -loop massless integrals up to $O(\varepsilon^0)$ with only one external momentum.

3.2 Full calculational procedure with a typical four-loop diagram

Now we turn to demonstrate a full calculational procedure for the typical four-loop diagram pictured in fig.1, which contributes to Z.

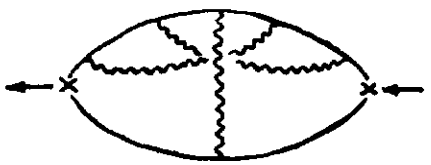


Fig.1 *A typical four-loop nonplanar type diagram*

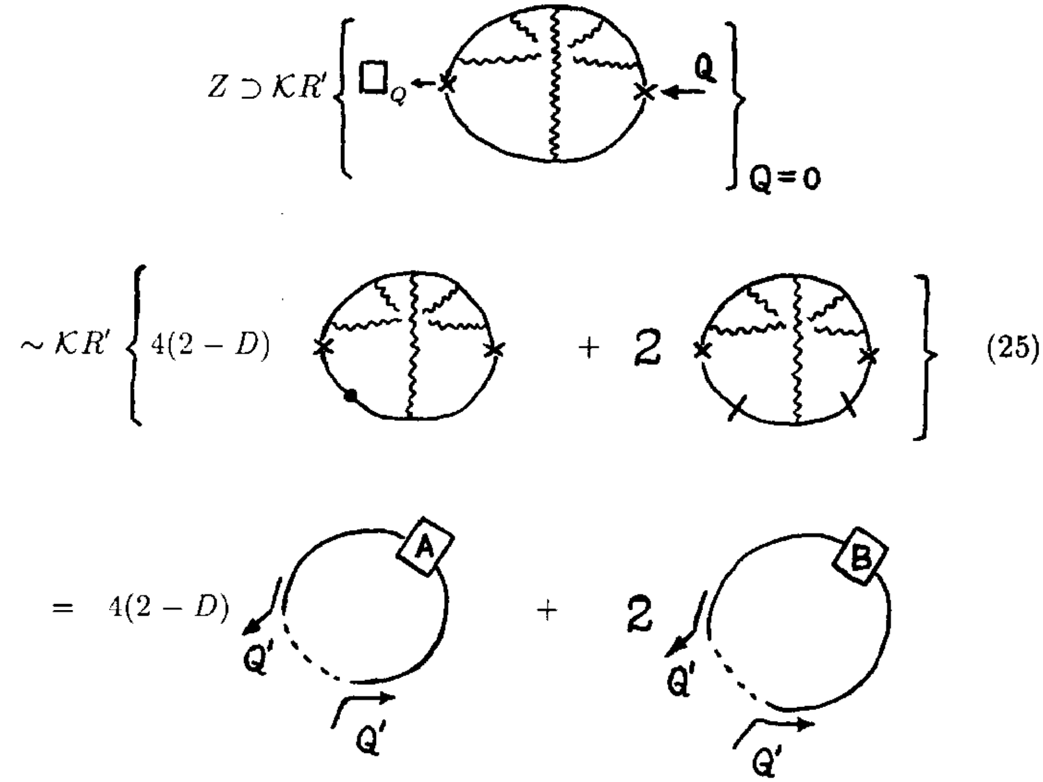
We need to evaluate the counterterm to the diagram pictured in fig.1. In other words, we have to evaluate $-\mathcal{K}R'$ of this diagram. It is easy to see that the given diagram diverges as

$$G \sim \lim_{Q\to\infty} Q^{4D-14}$$

and the superficial degree of divergence $d = 2$. Now, using the fact that the counterterm has only polynomial dependence on the external momenta Q within the minimal subtraction prescription, one can remove such a dependence by differentiating twice the diagram with respect to Q and then set external momenta to zero. At the next step, because there is no more dependence on the external momenta, one can introduce a new fictitious external momentum, flowing through one of the lines. This

[2]The new program HEPLoops (version 3) [17] calculates the three-loop Feynman diagrams up to terms of $O(\varepsilon)$ and some types of four-loop diagrams up to their finite parts. Some of the four-loop diagrams contributing to $R(s)$ have been recalculated by HEPLoops.

line should be chosen, according to the necessity of symplifying the topology of the diagram, and to avoid infrared divergences.



where $\Box_Q = \partial^2/\partial Q_\mu \partial Q^\mu$. The dot and dashes on the lines result from differentiating the corresponding fermion lines:
$\Box_Q[\longleftarrow (P+Q)]_{Q=0} \equiv 2(2-D)[\leftarrow\!\bullet\!- P] = 2(2-D)\hat{P}/P^4$;
$\partial/\partial Q^\alpha[\longleftarrow (P+Q)]_{Q=0} \equiv [P \longleftarrow\!\!\!\!\!\!\!+\!\!\!\longleftarrow P] = (-\hat{P}/P^2)\gamma^\alpha(\hat{P}/P^2)$.
Boxes containing the corresponding three-loop propagator type insertions with all divergences, including the overall one, are subtracted (complete R - operation). The dotted lines means that this line is temporarily "torn". After evaluation of boxes the parts of the torned line should be pasted together again and the fourth loop integration should be done, taking into account the corresponding exponents of the propagators due to the three-, two- and one-loop "box" insertions. The above procedure gives a large simplification of the problem. Indeed, at the next step we can calculate the three-loop propagator type insertions (boxes) and, finally, evaluate the fourth trivial loop.

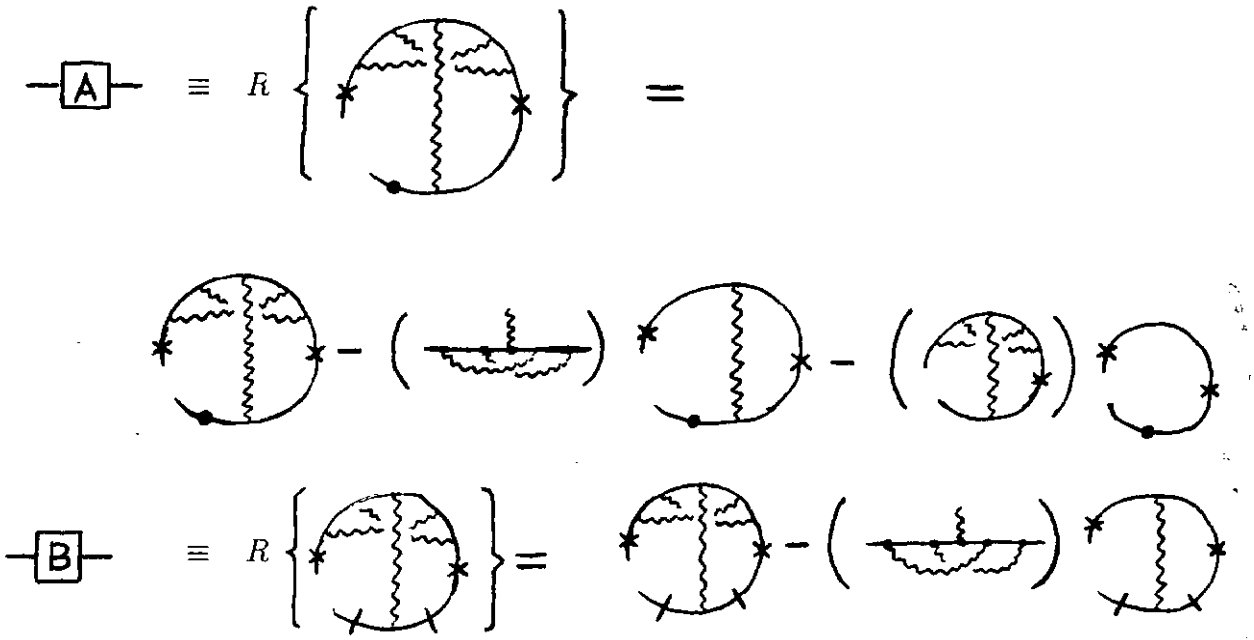


Fig.2 *Complete R-operation for the 3-loop insertions*

The complete R-operation of the three-loop diagram insertion corresponding to the ones at the r.h.s. of (25) is given in fig.2. Graphs in the brackets correspond to two- and three-loop cunterterms respectively. Now, the problem of calculating the four-loop contribution to Z is reduced to evaluation of the one-, two- and three-loop propagator type massless diagrams up to $O(\varepsilon^0)$. These calculations can be done by using the program MINCER [15] and LOOPS [14]. Note, that in the present calculation we have used the extended by us version of MINCER, which, in particular, includes the new block for 4th loop integrations and the block for ultraviolet renormalizations. We have used also the PC version of LOOPS for calculating one and two-loop counterterms (see more details below).

It is important to stress that, in fact, it is sufficient to evaluate only the $\mathcal{K}R'$ for the relevant three-loop insertion. In other words, there is no necessity to calculate three-loop counterterms. Indeed, the more detailed analysis gives:

$$R[G] = R'[G] - (1 - D/2)\mathcal{K}\left(\frac{1}{1 - D/2}R'[G]\right) \tag{26}$$

where G is the corresponding three-loop insertion. The relation (26) allows simple computer implementations and facilitates calculations considerably.

Complete R-operation of each four-loop diagram generally has the form:

$$\left(\frac{\mu^2}{Q^2}\right)^{4\varepsilon} f_4(\varepsilon) - \sum_{l=1}^{3}\left(\frac{\mu^2}{Q^2}\right)^{(4-l)\varepsilon} c_l(1/\varepsilon) f_{4-l}(\varepsilon)$$

where, $f_i(\varepsilon)$ is the result of calculation of the corresponding Feynman graph including the last trivial loop integration. c_l are the l-loop counterterms (polynomial in $1/\varepsilon$). According to the well known Bogoliubov theorem [1], the terms of the type $1/\varepsilon^n(ln\mu^2/Q^2)^m$, which appear due to the expansion of the factors $(\mu^2/Q^2)^{l\varepsilon}$ into the Laurent series in ε, must be canceled in the final answer for the particular diagram. This fact gives us an opportunity to test the calculations during the evaluation of each diagram.

Finally, for the contribution to the Z of the diagram pictured on fig.2 we obtain:

$$\left(\frac{\alpha_s}{4\pi}\right)^3 N_F C_F(C_F - C_A)(C_F - C_A/2) \left[+4\frac{1}{\varepsilon^3} - 26\frac{1}{\varepsilon^2} + \frac{65}{4}\frac{1}{\varepsilon} - 40\zeta(3)\frac{1}{\varepsilon^3}\right]$$

The CPU time for the above diagram on the 0.8 MFlop EC-1037 IBM compatible mainframe was over 6 hours. The extended version of the program MINCER for the system SCHOONSCHIP was used.

3.3 Four-loop results

The total number of topologically distinct Feynman diagrams contributing to $Z_{1,i}$ is 1; to $Z_{2,i}$ is 2; to $Z_{3,i}$ is 17 and to $Z_{4,i}$ is 98. However, after application of the IR-procedure, which as discussed above, involves differentiation twice with respect to external momentum of the diagram, the number of four-loop diagrams, which need to be calculated, increases to approximately 250. Furthermore, there are one-, two- and three-loop diagrams, approximately 600, which need to be calculated to subtract subdivergences (evaluate R') for all four-loop diagrams.

All analytical calculations of the four-loop diagrams have been done by using the program, which is an extended version of the program MINCER [15]. Evaluation of one- and two-loop counterterms has been performed by the PC version of the program LOOPS [14]. The above programs are written on the algebraic programming systems SCHOONSCHIIP [19] and REDUCE [18] respectively. We have used three IBM compatible, 0.8MFlop EC-1037 mainframes with SCHOONSCHIP system and IBM/PC AT-386 24MHz computer with the REDUCE system. The total CPU time for the mainframes was over 700 hours. Later, some of the difficult four-loop diagrams have been recalculated by the HEPLoops - new program for analytical multiloop calculations [17], using only the IBM/PC AT computer. This program is written on the algebraic programming system FORM [20], and is 3-7 times more efficient then the program MINCER (depending on the type of computer and the problem complexity). Program HEPLoops is especially oriented towards very large scale four-loop calculations.

The full graph-by-graph results will be given elsewhere. The Feynman gauge is used. The momentum integrations are performed within the $\overline{MS}$ prescription, which

amounts to formally setting $\gamma = \zeta(2) = ln4\pi = 0$. We obtain the following analytical result for the four-loop photon renormalization constant in QCD:

$$
\begin{aligned}
&Z_{ph} = 1 - Z = \\
&\quad 1 + 3\sum_f Q_f^2 \left[-\frac{4}{3}\frac{1}{\varepsilon} + \frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon}(-2C_F)\right)\right. \\
&\quad + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{\varepsilon^2}\left(\frac{22}{9}C_F C_A - \frac{8}{9}N_f T C_F\right) + \frac{1}{\varepsilon}\left(\frac{2}{3}C_F^2 - \frac{133}{27}C_F C_A + \frac{44}{27}N_f T C_F\right)\right] \\
&\quad + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[\frac{1}{\varepsilon^3}\left(-\frac{121}{27}C_F C_A^2 + \frac{88}{27}N_f T C_F C_A - \frac{16}{27}N_f^2 T^2 C_F\right)\right. \\
&\quad + \frac{1}{\varepsilon^2}\left(-\frac{11}{9}C_F^2 C_A + \frac{2381}{162}C_F C_A^2 - \frac{14}{9}N_f T C_F^2 - \frac{778}{81}N_f T C_F C_A + \frac{88}{81}N_f^2 T^2 C_F\right) \\
&\quad + \frac{1}{\varepsilon}\left(\frac{23}{2}C_F^3 + \left(-\frac{430}{27} + \frac{88}{9}\zeta(3)\right)C_F^2 C_A + \left(-\frac{5815}{972} - \frac{88}{9}\zeta(3)\right)C_F C_A^2\right. \\
&\quad \left.\left.\left. + \left(\frac{338}{27} - \frac{176}{9}\zeta(3)\right)N_f T C_F^2 + \left(\frac{769}{243} + \frac{176}{9}\zeta(3)\right)N_f T C_F C_A + \frac{308}{243}N_f^2 T^2 C_F\right)\right]\right] \qquad (27) \\
&\quad + \left(\frac{\alpha_s}{4\pi}\right)^3 \left(\sum_f Q_f\right)^2 \left(\frac{d^{abc}}{4}\right)^2 \left(-\frac{176}{9} + \frac{128}{3}\zeta(3)\right)\frac{1}{\varepsilon}
\end{aligned}
$$

It should be stressed that the Riemann ζ-functions $\zeta(4)$ and $\zeta(5)$, which appear at the graphwise level are canceled in the above expression. Moreover, as we have observed, the $\zeta(4)$ has disappeared within each gauge invariant set of diagrams. It is easy to see that $\zeta(3)$ disappears for QED ($C_F = 1, C_A = 0, T = 1/2$) except the last term, which comes from the "light-by-light" type diagrams (fig.3). Terms of such type appear only at the four-loop level and in higher-order. We still have no explanation for the cancellation of the ζ-functions. Note that the diagrams pictured in fig.3 are the most complicated diagrams and the computation of each of them requires over 80h of CPU time. Note also that the result (27) does not depends on the choice of scheme within the MS-type schemes.

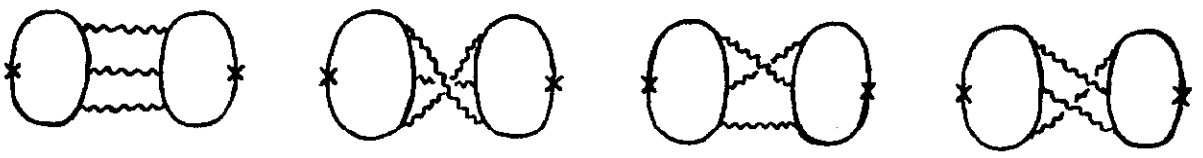


Fig.3 *"Light-by-light" type diagrams*

In order to evaluate $R(s)$ to $O(\alpha_s^3)$, besides the four-loop Z we have to calculate the bare quantity $\Pi^B(Q^2)$ to the three-loop level, keeping the terms of $O(\varepsilon)$ for the two-loop and terms of $O(\varepsilon^0)$ for the three-loop results. We get the following analytical result for the unrenormalized hadronic vacuum polarization function at the three-loop level in the $\overline{MS}$ scheme:

$$
\begin{aligned}
&\Pi^B(Q^2) = \\
&\quad 3\sum_f Q_f^2 \Bigg[\left(\frac{4}{3}\frac{1}{\varepsilon} + \frac{20}{9} + \frac{112}{27}\varepsilon + \frac{656}{81}\varepsilon^2 - \frac{28}{9}\zeta(3)\varepsilon^2\right) \\
&\quad + \left(\frac{\alpha_s}{4\pi}\right) C_F \left(2\frac{1}{\varepsilon} + \frac{55}{3} - 16\zeta(3) + \varepsilon\left(\frac{1711}{18} - \frac{152}{3}\zeta(3) - 24\zeta(4)\right)\right) \\
&\quad + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(C_F^2\left(-\frac{2}{3}\frac{1}{\varepsilon} - \frac{286}{9} - \frac{296}{3}\zeta(3) + 160\zeta(5)\right)\right. \\
&\quad + C_F C_A \left(\frac{44}{9}\frac{1}{\varepsilon^2} + \frac{1948}{27}\frac{1}{\varepsilon} - \frac{176}{3}\zeta(3)\frac{1}{\varepsilon} + \frac{50339}{81} - \frac{3488}{9}\zeta(3) - 88\zeta(4) - \frac{80}{3}\zeta(5)\right) \\
&\quad \left. + C_F T N_f \left(-\frac{16}{9}\frac{1}{\varepsilon^2} - \frac{704}{27}\frac{1}{\varepsilon} + \frac{64}{3}\zeta(3)\frac{1}{\varepsilon} - \frac{17668}{81} + \frac{1216}{9}\zeta(3) + 32\zeta(4)\right)\right) + O(\alpha_s^3)\Bigg]
\end{aligned}
\tag{28}
$$

Substituting the expressions for the relevant $Z_{i,j}$ and $\Pi_{i,j}$ into eq. (24) and recalling the values for β_0 and β_1, we get the following analytical result for $R(s)$ at the four-loop level, in the $\overline{MS}$-scheme:

$$
\begin{aligned}
&R^{\overline{MS}}(s) = \\
&\quad 3\sum_f Q_f^2 \left(1 + \left(\frac{\overline{\alpha_s}}{4\pi}\right)(3C_F)\right. \\
&\quad + \left(\frac{\overline{\alpha_s}}{4\pi}\right)^2 \left[C_F^2\left(-\frac{3}{2}\right) + C_F C_A\left(\frac{123}{2} - 44\zeta(3)\right) + N_f T C_F(-22 + 16\zeta(3))\right] \\
&\quad + \left(\frac{\overline{\alpha_s}}{4\pi}\right)^3 \left[C_F^3\left(-\frac{69}{2}\right) + C_F^2 C_A(-127 - 572\zeta(3) + 880\zeta(5))\right. \\
&\quad + C_F C_A^2\left(\frac{90445}{54} - \frac{10948}{9}\zeta(3) - \frac{440}{3}\zeta(5)\right) \\
&\quad + N_f T C_F^2(-29 + 304\zeta(3) - 320\zeta(5)) \\
&\quad + N_f T C_F C_A\left(-\frac{31040}{27} + \frac{7168}{9}\zeta(3) + \frac{160}{3}\zeta(5)\right) \\
&\quad \left. + N_f^2 T^2 C_F\left(\frac{4832}{27} - \frac{1216}{9}\zeta(3)\right) - \pi^2 C_F\left(\frac{11}{3}C_A - \frac{4}{3}N_f T\right)^2\right] + O(\alpha_s^4)\Bigg) \\
&\quad + \left(\frac{\overline{\alpha_s}}{4\pi}\right)^3 \left(\sum_f Q_f\right)^2 \left(\frac{d_{abc}}{4}\right)^2 \left[\frac{176}{3} - 128\zeta(3)\right] + O(\alpha_s^4)
\end{aligned}
\tag{29}
$$

Note, that $\zeta(5)$ appears in the final result due to the contributions from $\Pi_{3,0}$. For the standard QCD with the color $SU_c(3)$ gauge group we obtain:

$$
\begin{aligned}
&R^{\overline{MS}}(s) = \\
&\quad 3\sum_f Q_f^2 \left(1 + \left(\frac{\overline{\alpha_s}}{\pi}\right) + \left(\frac{\overline{\alpha_s}}{\pi}\right)^2 \left[\frac{365}{24} - 11\zeta(3) - N_f\left(\frac{11}{12} - \frac{2}{3}\zeta(3)\right)\right]\right. \\
&\quad + \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 \left[\frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5)\right. \\
&\quad + N_f\left(-\frac{7847}{216} + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right) + N_f^2\left(\frac{151}{162} - \frac{19}{27}\zeta(3)\right) \\
&\quad \left.\left. - \frac{\pi^2}{48}\left(11 - \frac{2}{3}N_f\right)^2\right]\right) + \left(\sum_f Q_f\right)^2 \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 \left[\frac{55}{72} - \frac{5}{3}\zeta(3)\right] + O(\alpha_s^4)
\end{aligned}
\tag{30}
$$

and finally, taking into account the values for the relevant Rieman ζ -functions: $\zeta(3) = 1.2020569$ and $\zeta(5) = 1.0369278$ we obtain the numerical form:

$$
\begin{aligned}
&R^{\overline{MS}}(s) = \\
&\quad 3\sum_f Q_f^2 \left[1 + \left(\frac{\overline{\alpha_s}}{\pi}\right) + \left(\frac{\overline{\alpha_s}}{\pi}\right)^2 (1.986 - 0.115N_f)\right. \\
&\quad \left. + \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 (-6.637 - 1.200N_f - 0.005N_f^2)\right] - \left(\sum_f Q_f\right)^2 \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 1.240 + O(\alpha_s^4)
\end{aligned}
\tag{31}
$$

The obtained result is smaller by an order of magnitude and has the opposite sign compared to the results of [25]. The terms of order $O(N_f)$ and $O(N_f^2)$ coincide with the corresponding terms of the previous result of [25]. Note that only 23 four-loop diagrams contribute to $O(N_f)$ and 2 four-loop diagrams contribute to $O(N_f^2)$. These diagrams are much simpler then the other ones, since one can directly substitute the results for one- or two-loop fermion loop insertions. The most complicated diagrams are pictured in fig.4. The CPU time for each of them was over 100h and the intermediate expression had as many as $\sim 10^5 - 10^6$ terms.

fig.4 *Some of the most complicated diagrams*

There are at least two sources of error in the program [44], which was used in the previous calculation [25]. The first is an incorrect trace calculation and the second is due to an error in the subroutine, which calculates the expansions of the Laurent series in ε of the basic one-loop integrals. Note that the algorithm [10] reduces all multiloop calculations to the calculation of some basic one-loop integrals (the so called G-functions). Unfortunately, both errors affected the terms of order $O(\varepsilon^0)$ only and could not be eliminated by the tests of cancellations of poles, including those proportional to logarithmic terms. More details on the discovering of errors in the program [44] are given in sect. of concluding remarks.

It is known, that the perturbative coefficients for $R(s)$ are scheme dependent. The above result was obtained in the modified minimal subtraction, the so-called $\overline{MS}$-scheme [32]. Here we present the results of calculation in another version of minimal subtraction scheme, the so called G-scheme, which was introduced in [10] and is very convenient for practical calculations. The G-scheme is defined in such a way that the trivial one-loop integral in this scheme is:

$$\int \frac{d^D P}{(2\pi)^D} \frac{1}{P^2 (P-K)^2} = \frac{1}{4\pi^2} \frac{1}{K^{2(2-D/2)}} \frac{1}{\varepsilon}$$

and the result for $R(s)$ looks like:

$$R^G(s) = 3 \sum_f Q_f^2 \left[1 + \left(\frac{\overline{\alpha_s}}{\pi}\right) + \left(\frac{\overline{\alpha_s}}{\pi}\right)^2 (-3.514 + 0.218 N_f) + \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 (-10.980 - 0.692 N_f + 0.03 N_f^2) \right] - \left(\sum_f Q_f\right)^2 \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 1.240 + O(\alpha_s^4) \qquad (32)$$

Finally, in the original MS-scheme [9] we get:

$$R^{MS}(s) =$$
$$3\sum_f Q_f^2 \left[1 + \left(\frac{\overline{\alpha_s}}{\pi}\right) + \left(\frac{\overline{\alpha_s}}{\pi}\right)^2 (7.359 - 0.326N_f)\right.$$
$$\left. + \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 (56.026 - 8.779N_f + 0.176N_f^2)\right] - \left(\sum_f Q_f\right)^2 \left(\frac{\overline{\alpha_s}}{\pi}\right)^3 1.240 + O(\alpha_s^4) \tag{33}$$

The analysis of the problem of the scheme dependence of the results will be given in sect.6.

4 The $O(\alpha_s^3)$ Contribution to $\Gamma(\tau^- \to \nu_\tau + hadrons)$

Another important process for testing of the standard model predictions, is the hadronic decay of the τ - lepton.

As was shown in [29, 30], the ratio

$$R_\tau = \frac{\Gamma(\tau^- \to \nu_\tau + hadrons)}{\Gamma(\tau^- \to \nu_\tau e^- \overline{\nu}_e)} \tag{34}$$

is calculable in perturbative QCD. Strictly speaking, besides the QCD perturbative parts the nonperturbative and weak contributions should be included to estimate R_τ:

$$R_\tau = R^\tau_{pert} + R^\tau_{nonpert} + R^\tau_{weak} \tag{35}$$

where, each of terms in the r.h.s. was estimated in [29, 30]. However, in calculations of $O(\alpha_s^3)$ corrections to the R^τ_{pert} the incorrect value for the four-loop correction to $R(s)$ [25] was used. Here we use the new value for the four-loop $R(s)$ (see eq.(29-31)) and obtain a corrected expression for the four-loop R^τ_{pert}. (A short presentation of this result has been made in [31].)

4.1 Perturbative QCD contributions

The perturbative part of the ratio R_τ can be expressed by the integral around a contour $\mid s \mid = M_\tau^2$ in the complex s-plane (fig.5). In the chiral limit it looks like (See

[29, 30].):

$$R^T_{pert} = 6\pi i \int_C \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^T(s)\right] \tag{36}$$

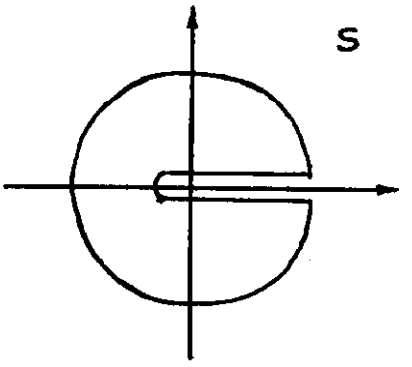


Fig.5 *Integration contour C*

The transverse part of the relevant combination of vector and axial vector color singlet quark current two-point correlation functions $\Pi^T(s)$ is related to the $D(s)$-function, which was defined in sect.1:

$$s\frac{d}{ds}\Pi^T(s) = \frac{3\sum \mid V_{ff'} \mid^2}{3\sum_f Q_f^2} D(s) \tag{37}$$

where the $V_{ff'}$ are the Kobayashi-Maskawa matrix elements. Reexpanding the $\alpha_s(s)$ in the power series of $\alpha_s(M_\tau^2)$, we obtain:

$$\begin{aligned}\frac{\alpha_s(s)}{\pi} &= \frac{\alpha_s(M_\tau^2)}{\pi} + \left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^2 \frac{\beta_0}{2} ln\frac{s}{M_\tau^2} \\ &+ \left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^3 \left(\frac{\beta_1}{2} ln\frac{s}{M_\tau^2} + \frac{\beta_0^2}{4} ln^2\frac{s}{M_\tau^2}\right) + O(\alpha_s^4)\end{aligned} \tag{38}$$

Performing the integration in (36) and taking into account (38) we obtain in terms of perturbative coefficients of $R(s)$:

$$R_\tau^{pert} = 3(\mid V_{ud} \mid^2 + \mid V_{us} \mid^2) \left[R_0 + \frac{\alpha_s(M_\tau^2)}{4\pi} R_1 + \left(\frac{\alpha_s(M_\tau^2)}{4\pi}\right)^2 \left(R_2 - \frac{19}{3}\beta_0\right) \right.$$
$$\left. + \left(\frac{\alpha_s(M_\tau^2)}{4\pi}\right)^3 \left(R_3 - \frac{19}{6}R_2\beta_0 - \frac{19}{3}\beta_1 + \left(\frac{265}{18} - \frac{4\pi^2}{3}\right)\beta_0^2\right) + O(\alpha_s^4)\right] \quad (39)$$

Note that the diagrams pictured in fig.3 do not contribute to R_τ. So, the term which is proportional to the $(\sum_f Q_f)^2$ should be omitted.

The analytical result for the ratio R_τ in the $\overline{MS}$-scheme with $(\mid V_{ud} \mid^2 + \mid V_{us} \mid^2) \approx 1$ looks like:

$$R_\tau^{pert} =$$
$$3\left(1 + \left(\frac{\overline{\alpha_s}(M_\tau)}{\pi}\right) + \left(\frac{\overline{\alpha_s}(M_\tau)}{\pi}\right)^2 \left[\frac{313}{16} - 11\zeta(3) - N_f\left(\frac{85}{72} - \frac{2}{3}\zeta(3)\right)\right]\right.$$
$$+ \left(\frac{\overline{\alpha_s}(M_\tau)}{\pi}\right)^3 \left[\frac{544379}{1152} - \frac{8917}{24}\zeta(3) + \frac{275}{6}\zeta(5)\right. \quad (40)$$
$$+N_f\left(-\frac{8203}{144} + \frac{733}{18}\zeta(3) - \frac{25}{9}\zeta(5)\right) + N_f^2\left(\frac{3935}{2592} - \frac{19}{18}\zeta(3)\right)$$
$$\left.\left. -\frac{\pi^2}{48}\left(11 - \frac{2}{3}N_f\right)^2\right] \quad + \quad O(\alpha_s^4)\right)$$

or, substituting relevant values for Rieman ζ-functions and π we obtain:

$$R_\tau^{pert} =$$
$$3\left[1 + \left(\frac{\overline{\alpha_s}(M_\tau)}{\pi}\right) + \left(\frac{\overline{\alpha_s}(M_\tau)}{\pi}\right)^2 (6.3399 - 0.3792N_f)\right. \quad (41)$$
$$\left. + \left(\frac{\overline{\alpha_s}(M_\tau)}{\pi}\right)^3 (48.5832 - 7.8795N_f + 0.1579N_f^2) + O(\alpha_s^4)\right]$$

Note, that this result is more sensitive to the number of quark flavours than the result for e^+e^- annihilation. An analogous calculation has been done in ref. [45], the result of which agrees with ours [31]. These results were confirmed in ref.[28].

4.2 Nonperturbative and Electroweak contributions.

The nonperturbative contributions to R_τ can be expressed as a series of power corrections in $1/M_\tau^2$:

$$R_\tau^{nonpert} \sim \frac{C_2 f(m_q^2(M_\tau), \theta_c)}{M_\tau^2} + \sum_{i \geq 2} \frac{C_{2i} < O_{2i} >_0}{M_\tau^{2i}} \tag{42}$$

where the m_q are u, d, s running quark masses, $< O_{2i} >_0$ are the so-called vacuum condensates, which are usually evaluable phenomenologically and the C_i are their coefficient functions, which include short distance effects. Note, that nowadays the only way to estimate the strong interaction contributions of the condensates is a perturbation theory calculation. The high-order perturbative QCD contributions to the coefficient functions of the dimension 4 corrections (of the gluon $< \alpha_s G^2 >_0$ and quark $< m_f \bar{q}_f q_f >_0$ condensates) have been calculated in refs. [26, 46] and confirmed in ref. [47]. The leading perturbative QCD contributions to the coefficient functions of the dimension 6 corrections was calculated in ref. [48]. However, as was shown in ref. [26] high-order perturbative corrections to some of the coefficient functions are too large. For example, for the coefficient functions of the strange quark condensate in the vector channel $\Lambda_{eff} \approx 30 \Lambda_{\overline{MS}}$. This means that renormalization group invariant criteria to calculability of the QCD contributions to the coefficient function via perturbation theory is not fulfilled. The coefficient functions of the gluon and dimension 6 condensates are calculated up to $O(\alpha_s)$ order and to analyze the convergence we need the next two corrections [49]. The above situation allows us to estimate QCD corrections to the condensate contributions not better then the order of magnitude. On the other hand there is another source of theoretical uncertainties in estimation of condensate contributions. The situation is especially complicated at dimension 6 and higher, where the operator basis of expansion includes a large number of operators. Nowadays, there are no precise methods to estimate their matrix elements. For the matrix elements of four quark operators (dimension 6) the vacuum saturation approximation [6] should be used to express them as the square of the two-quark matrix element. Finally, the following should be stressed. In fact we are trying to estimate contributions from the series (power series) of the asymptotic series (coefficient functions). In this case, the unsuitable rule of inclusion of the high order corrections to the coefficient functions could lead to a deviation from the true value. (Such an effect was observed in ref. [49] for the QCD sum rule method.) So, the question of how many orders should be included for a particular coefficient function, could be answered only after the detailed investigation of each particular case.

Summarizing the above discussion, we conclude that due to the large theoretical uncertainties in the estimation of the nonperturbative contributions, their inclusion is meaningless, especially as the size of such contributions is definitely less then 1% (about $\pm 0.5\%$). The $O(m^2)$ order terms has a pure perturbative character. The

calculation of the coefficient functions of the $O(m^2)$ terms at the three-loop level has been done in [38] (However, the results of [38] are incorrect. These results has been corrected in [39].) Here we note that, in this case the perturbative series is also not well-behaved. Moreover, the contribution of such terms is suppressed by $sin^2\theta_c$ and even for the strange quark case is less than 1%. So, in further estimations we neglect all power corrections and estimate their contribution in the total theoretical uncertainties in the evaluation of R_τ as a $\pm 1\%$.

The leading order electroweak corrections were calculated in ref. [50]. These corrections give roughly +2% contributions to the R_τ. We will include these corrections in the multiplicative factor [50]: $\eta_{weak} = 1.0225 \pm 0.0050$. Recall, that we set $(\mid V_{ud} \mid^2 + \mid V_{us} \mid^2) \approx 1$.

5 Four-loop QED Renormalization Group Functions

We now turn to the calculation of the standard QED renormalization group functions. These quantities can be obtained as an intermediate result of the calculations of $R(s)$, replacing the $SU_c(3)$ gauge group invariants for the corresponding diagrams in a proper way.

5.1 General formulae

The Lagrangian density of standard QED is:

$$L_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\sum_j \overline{\psi}_j\gamma^\mu D_\mu\psi_j - \sum_j m_j\overline{\psi}_j\psi_j - \frac{1}{2\alpha_G}\partial_\mu A^\mu\partial_\mu A^\mu \qquad (43)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$. α_G is the gauge parameter, m_j are the fermion masses, ψ and A_μ are the fermion and photon fields and e is the electric charge.

Renormalization constants are defined by the relations:

$$\begin{aligned} \psi_B &= Z_2\psi_R \\ A^\mu_B &= Z_{ph}^{1/2}A^\mu_R \\ \alpha_B &= Z_\alpha\alpha_R \qquad (\alpha = e^2/4\pi) \end{aligned} \qquad (44)$$

It is easy to obtain the Ward identity in the following form:

$$Z_\alpha = Z_{ph}^{-1} \qquad (45)$$

which leads to the relation:

$$Z_{ph}\alpha_B = \alpha_R \tag{46}$$

Using relation (46) and the standard definition of the QED β-function within the MS-scheme:

$$\frac{1}{4\pi}\beta_{QED}(\alpha) = \frac{1}{4\pi}\mu^2\frac{d\alpha}{d\mu^2}\mid_{\mu^{2\epsilon}\alpha_B - fix} + \alpha\epsilon \tag{47}$$

we obtain our main formula:

$$\beta_{MS}(\alpha) = -\frac{1}{4\pi}\lim_{\epsilon\to 0}\frac{\epsilon\alpha}{1-\alpha\frac{\partial}{\partial\alpha}ln Z_{ph}} \tag{48}$$

5.2 Four-loop results

The photon field renormalization constant Z_{ph} can be found from the QED relation (analogous to (27)), where only 58 QED four-loop diagrams are contributing to the $\Pi(\mu^2/Q^2,\alpha)$. The prescription for the evaluation of the diagram contributions to the Π_B is analogous to the case described in sect.2. The total CPU time on the three IBM compatible EC-1037 computers was approximately 400 hours.

Using relations (18) and (48) and the relevant graph-by-graph results for the QED case ($C_F = 1$, $C_A = 0$, $T = 1$), we obtain the following result for the four-loop QED β- function in the MS-type schemes:

$$\begin{aligned}\beta_{QED}(\alpha) = &\frac{4}{3}N_f\left(\frac{\alpha}{4\pi}\right)^2 + 4N_f\left(\frac{\alpha}{4\pi}\right)^3 - N_f\left(2+\frac{44}{9}N_f\right)\left(\frac{\alpha}{4\pi}\right)^4 \\ &-N_f\left[46-\left(\frac{760}{27}-\frac{832}{9}\zeta(3)\right)N_f+\frac{1232}{243}N_f^2\right]\left(\frac{\alpha}{4\pi}\right)^5 + O(\alpha^6)\end{aligned} \tag{49}$$

The four-loop QED β- function first was calculated in [51] with the help of the program [44]. However, further consideration shows that the results of [51] are wrong. The four-loop part of our result is clearly different (and are simpler in appearance than) the result in [51]. The same result was obtained also by the authors of ref. [51]. The above result was reported in the joint talk [52] and published in refs. [27, 53].

It is useful for further applications to present the result for the F_1 function, which can be obtained from the result for β_{QED}, by subtracting the contributions of the diagrams with inside fermion loop insertions into the photon lines and reducing the power in $\alpha/4\pi$ by one (see, e.g. [1]). We obtain:

$$F_1(\alpha) = \frac{4}{3}\left(\frac{\alpha}{4\pi}\right) + 4\left(\frac{\alpha}{4\pi}\right)^2 - 2\left(\frac{\alpha}{4\pi}\right)^3 - 46\left(\frac{\alpha}{4\pi}\right)^4 \tag{50}$$

Note, that all coefficients up to four-loop level are rational numbers. In order to obtain the Gell-Man Low $\Psi(\alpha)$ function (QED β-function in the MOM-scheme) the

following transformation equation should be used (see: [1, 52, 53]):

$$\Psi(\alpha) = \beta(\alpha_{MS})\frac{\partial\alpha(\alpha_{MS})}{\partial\alpha_{MS}} \tag{51}$$

For the Ψ function at the four-loop level we obtain [52, 53]:

$$\begin{aligned}\Psi(\alpha) = &\frac{4}{3}N_f\left(\frac{\alpha}{4\pi}\right)^2 + 4N_f\left(\frac{\alpha}{4\pi}\right)^3 - N_f\left(2 + \left(\frac{184}{9} - \frac{64}{3}\zeta(3)\right)N_f\right)\left(\frac{\alpha}{4\pi}\right)^4 \\ &-N_f\left[46 - \left(104 + \frac{512}{3}\zeta(3) - \frac{1280}{3}\zeta(5)\right)N_f - \left(128 - \frac{256}{3}\zeta(3)\right)N_f^2\right]\left(\frac{\alpha}{4\pi}\right)^5 \\ &+O(\alpha^6)\end{aligned} \tag{52}$$

6 Renormalization Group Ambiguity of Perturbative QCD Predictions

Due to the key role of the e^+e^- annihilation R-ratio for the phenomenological consequences of the Standard Model, it is important to evaluate this quantity with sufficient precision theoretically. The accuracy of present experimental measurements of the QCD effects in R at e^+e^- - colliders is about 13-15% [2]. However, the only way to theoretically estimate $R(s)$ is by perturbation theory, which involves calculation of a large number of Feynman diagrams and, as we have discussed above, requires a huge amount of computer and human resources. For example, to $O(\alpha_s^3)$ we have calculated 98 four-loop Feynman diagrams. [27]. The next order requires calculation of approximately 600-700 five-loop diagrams. Calculations of such a scale are extremely difficult. On the other hand, perturbative QCD series are asymptotic ones and the question - how many orders need to be calculated, could be answered only from estimations of remainders (see, e.g., [7]). Moreover, perturbative coefficients beyond the two-loop level, as well as the expansion parameter, are scheme-scale dependent. The scheme-scale ambiguity - fundamental property of the renormalization group calculations in QCD - doesn't allow one to obtain reliable estimates from the first few calculated terms without involving additional criteria.

In this section we discuss the problem of extraction of reliable estimates for observable quantities via perturbation theory. The problem of scheme-scale dependence of perturbative QCD predictions will be considered. We will apply the three most effective approaches for resolving the scheme-scale ambiguity and we will fix the scale for which all of the criteria tested are satisfied for the quantity $R(s)$ at the four-loop level. The quantity R_τ will also be considered. Note, that we will deal only with one parametric family of the MS-type schemes.

6.1 Perturbative QCD series

The R-ratio is given by the perturbative calculation in the following form:

$$R(s) = r_0 \left(1 + r_1(\frac{s}{\mu^2})\frac{\alpha_s(\mu)}{\pi} + r_2(\frac{s}{\mu^2})(\frac{\alpha_s(\mu)}{\pi})^2 + r_3(\frac{s}{\mu^2})(\frac{\alpha_s(\mu)}{\pi})^3 + ...\right) \qquad (53)$$

We are considering high enough energies where R is a function of a single variable - the center-of-mass energy squared (massless case). Our aim is to evaluate pure QCD effects in R, which start with the term $O(\alpha_s)$. Without losing generality, we adopt 't Hooft's "minimal subtraction" (MS) prescription [9]. There is an ambiguity in the choice of renormalization scale parameter μ. Usually we set $\mu^2 = s$ and absorb the large log's in the definition of the running coupling. On the other hand the choice $\mu^2 = \chi s$ ($\chi \equiv e^{-t}$) for all χ gives equivalent expansions. Obviously the sum of "all" terms in (53) doesn't depend on the choice of μ. However, in practice, we deal with truncated series, where the sum has a nontrivial dependence on the choice of renormalization parameter. Here we will keep the "natural" choice $\mu^2 = s$ and the ambiguity will transfer to the prescription: $\int d^4p \longrightarrow \int d^{4-2\epsilon}p(\mu^2 e^{(-t+O(\epsilon))})^\epsilon$. By changing t we get different MS-type schemes. We can always reexpand (53) in a new scheme (with a new Λ in (5)) and so redistribute the values of r_i ($i > 1$). All these schemes are equivalent. On the other hand a new scheme may be "better," but we can conclude this only based on the knowledge of remainders. The problem of scheme-scale ambiguity, which is in fact a problem of remainders can be formulated as follows: How does one choose the scheme (or Λ) in order to make remainder minimal in the series of type (53) for the given range of energy and what is the numerical uncertainty of the approximation (53)? Here one should distinguish the following two questions. First, what is the best accuracy to which the given quantity is calculable via perturbation theory? Second, what is the accuracy of the given approximation? A few notes are in order. First, it should be stressed that perturbative QCD series are asymptotic and are apparently divergent. Moreover, no reliable estimates of the remainders are known at present. However, it is known from the theory of asymptotic series (see, e.g., [54]), that

$$|\sum_{i=1}^{N} r_i\alpha^i(Q^2) - R(Q^2)| = R_N \rightarrow \Delta R_{min}, \ \ when \ \ N \rightarrow N_{opt} \qquad (54)$$

which means that, the remainder R_N goes to its minimal value ΔR_{min} when the number of orders goes to its optimal value N_{opt}. Inclusion of the next to N_{opt} orders will lead away from the correct value. It is known that for a signchangable asymptotic series the remainder can be estimated by the first neglected term [54] (or by the last included term). However, it's still unknown if the QCD perturbative series has this character. We will assume as a hypothesis that within QCD one can estimate the remainder by the first neglected term or last included term. Now, the minimal

possible error, which defines the best accuracy of the perturbation theory for the given quantity has an order of $\Delta R_{min} \sim r_{N+1}\alpha^{N+1}(s)$. Note that both the number N_{opt} and the value of the ΔR_{min} depend on the range of energy.

6.2 Expression for $R(s)$ and R_τ within one parametric family of the MS-type schemes

Here, using the results of sect.3 we obtain the four-loop analytical result for $R(s)$ with explicit scale dependent form:

$$
\begin{aligned}
R^t(s) = \\
&3\sum_f Q_f^2 \left[1 + \left(\frac{\alpha_s^t}{\pi}\right)\right. \\
&+ \left(\frac{\alpha_s^t}{\pi}\right)^2 \left[\left(\frac{365}{24} - 11\zeta(3) + \frac{11}{4}t\right) + N_f\left(-\frac{11}{2} + \frac{2}{3}\zeta(3) - \frac{1}{6}t\right)\right] \\
&+ \left(\frac{\alpha_s^t}{\pi}\right)^3 \left[\left(\frac{87029}{288} - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5) - \frac{121}{48}\pi^2\right) + t\left(\frac{4321}{48} - \frac{121}{2}\zeta(3)\right) + t^2\frac{121}{16}\right. \\
&+N_f\left(\left(-\frac{7847}{216} + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta 5 + \frac{11}{36}\pi^2\right) + t\left(-\frac{785}{72} + \frac{22}{3}\zeta(3)\right) - t^2\frac{11}{12}\right) \\
&\left.\left.+N_f^2\left(\left(\frac{151}{162} - \frac{19}{27}\zeta(3) - \frac{1}{108}\pi^2\right) + t\left(\frac{11}{36} - \frac{2}{9}\zeta(3)\right) + \frac{1}{36}t^2\right)\right] + O(\alpha_s^4)\right] \\
&+ \left(\sum_f Q_f\right)^2 \left(\frac{\alpha_s^t}{\pi}\right)^3 \left[\frac{55}{72} - \frac{5}{3}\zeta(3)\right] + O(\alpha_s^4)
\end{aligned}
\tag{55}
$$

In numerical form we get:

$$
\begin{aligned}
R^t(s) = \\
&3\sum_f Q_f^2 \left[1 + \left(\frac{\alpha_s^t}{\pi}\right) + \left(\frac{\alpha_s^t}{\pi}\right)^2 [(1.9857 + 2.75t) - N_f(0.1153 + 0.1667t)]\right. \\
&+ \left(\frac{\alpha_s^t}{\pi}\right)^3 [(-6.6369 + 17.2964t + 7.5625t^2) - N_f(1.2001 + 2.0877t + 0.9167t^2) \\
&\left.+N_f^2(-0.0052 + 0.0384t + 0.0278t^2)]\right] - \left(\sum_f Q_f\right)^2 \left(\frac{\alpha_s}{\pi}\right)^3 1.2395 \; + \; O(\alpha_s^4)
\end{aligned}
\tag{56}
$$

where α_s^t can be parametrised in the form of (5) via $\Lambda_t = e^{-t/2}\Lambda_{\overline{MS}}$. Obviously, $t = 0$ corresponds to the $\overline{MS}$-scheme [32]. $t = ln4\pi - \gamma$ will transform the result to the original MS-scheme [9]. (Compare to eq.(29)-(31) and (33) respectively.) t=-2 corresponds to the G-scheme [43] (compare to eq.(32)).

For the perturbative contributions to the quantity R_τ we obtain an analogous expression:

$$
\begin{aligned}
&R_\tau^{pert}(t) = \\
&\quad 3\sum_f Q_f^2 \Bigg[1 + \left(\frac{\alpha_s^t}{\pi}\right) \\
&\quad + \left(\frac{\alpha_s^t}{\pi}\right)^2 \left[\left(\frac{313}{16} - 11\zeta(3) + \frac{11}{4}t\right) + N_f\left(-\frac{85}{72} + \frac{2}{3}\zeta(3) - \frac{1}{6}t\right)\right] \\
&\quad + \left(\frac{\alpha_s^t}{\pi}\right)^3 \left[\left(\frac{544379}{1152} - \frac{8917}{24}\zeta(3) + \frac{275}{6}\zeta(5) - \frac{121}{48}\pi^2\right) + t\left(\frac{3647}{32} - \frac{121}{2}\zeta(3)\right) + t^2\frac{121}{16}\right. \\
&\quad + N_f\left(\left(-\frac{8203}{144} + \frac{733}{18}\zeta(3) - \frac{25}{9}\zeta 5 + \frac{11}{36}\pi^2\right) + t\left(-\frac{497}{36} + \frac{22}{3}\zeta(3)\right) - t^2\frac{11}{12}\right) \\
&\quad \left.+ N_f^2\left(\left(\frac{3935}{2592} - \frac{19}{18}\zeta(3) - \frac{1}{108}\pi^2\right) + t\left(\frac{85}{216} - \frac{2}{9}\zeta(3)\right) + \frac{1}{36}t^2\right)\right] + O(\alpha_s^4)\Bigg]
\end{aligned}
\tag{57}
$$

and numerically:

$$
\begin{aligned}
R_\tau^{pert}(t) = 3\Bigg[&1 + \left(\frac{\alpha_s^t}{\pi}\right) + \left(\frac{\alpha_s^t}{\pi}\right)^2 [(6.3399 + 2.75t) - N_f(0.3792 + 0.1667t)] \\
&+ \left(\frac{\alpha_s^t}{\pi}\right)^3 [(48.5832 + 41.2443t + 7.5625t^2) - N_f(7.8795 + 4.9905t + 0.9167t^2) \\
&\qquad + N_f^2(0.1579 + 0.1264t + 0.0278t^2)] + O(\alpha_s^4)\Bigg]
\end{aligned}
\tag{58}
$$

6.3 Various approaches to resolve the scheme-scale ambiguity and the $\widetilde{MS}$-scheme

Several approaches were suggested to deal with the scheme-scale-remainder problem. Among them we will use the most effective ones:

1. "Effective scheme" approach [55], which is based on the principle of fastest convergence. Within this method the next to leading QCD correction should be absorbed in the definition of the "effective" running coupling. 2. Principle of "minimal sensitivity" [56]: If the approximant depends on "nonphysical parameters" (in our case μ or Λ), then these parameters should be chosen according to the criteria of minimal sensitivity of approximants to the variation of these parameters (assuming, of course, that the perturbative series are sumable). 3. Brodsky-Lepage-Mackenzie (BLM) approach [57], which suggests one fix the scale by the size of the quark vacuum polarization effects. As a result, the next to leading order QCD correction becomes independent of the number of quark flavors N_f. However, at higher orders there are different sources of N_f dependence (see below).

We will apply the above approaches to eq. (53) and will find a scale, which gives very good results for all considered criteria. (The results of this analysis has been briefly reported in ref. [58].) For the recent analysis see: [59] where the procedure of summing the perturbative series analytically continued from the euclidean region to Minkowski region has been suggested; The optimization of perturbation theory has been studied in ref. [60, 61] for jet cross sections in electron positron annihilation. The optimized perturbation theory is tested for different physical quantities in QED and QCD in [62]. In [63], the large order behavior of perturbation theory was studied. The scale ambiguity problem has been considered in [65], for the ϕ^3 model. In ref. [66], the attempt to extend of the BLM approach beyond the next-to-leading order has been made.

In fig.6 we have plotted $r_3(t)$ for different N_f. As one can see, within the region $t \in [-1.5, -0.5]$ r_3 has a very weak dependence on the number of flavors N_f as well as on the parameter t. Corresponding to the 3-loop coefficient $r_2(t)$ straight lines will intersect in one point for $t = -0.7 \in [-1.5, -0.5]$, which is obvious from the formula (55). The value $t = -0.7$ corresponds to the BLM result [57] $\mu^2 = \mu^2_{\overline{MS}} e^{0.710}$ and at this scale the flavor dependence is absorbed into the definition of the coupling. A few notes are in order. In the original BLM method [57] the scale is fixed by the contributions of quark vacuum-polarization insertions. At $O(\alpha_s^2)$ this is straightforward and corresponds to $t \approx -0.7$ [57]. However, in the expression (56) the term $O(N_f^2)$ is due purely to quark-vacuum polarization, while the $O(N_f)$ term has, in addition, contributions from the four diagrams with quark loop correction in the three gluon vertex. On the graph-by-graph level it is not difficult to separate the different type of contributions. We have analyzed the contributions of these diagrams

and found that their separation doesn't make a significant difference in the choice of scale.

In fig.7 we have plotted the dependence of the partial sums $R_n(t) = \sum_{m=1}^{n} r_m(t)(\alpha_s/\pi)^m$ on the parameter t. Here the parametrisation (5) was used and $ln\frac{s}{\Lambda^2} = 9$, $N_f = 5$. (The general picture doesn't change for other reasonable values of ln and N_f). We can see that Stevenson's "minimal sensitivity" principle [56] works perfectly for the wide range of $t \in [-1, +3]$ for four-loop approximant and $t \in [-2, 0]$ for three-loop approximant. (Similar analysis at three-loop level was done in [42]).

According to the above analysis we found that the scale $t = -0.710$ suggested in [57] is good at the four-loop level as well (fig.6) and this value is within minimal sensitivity region (fig.7). Moreover, we found that if we choose the t-parameter in the following analytical form: $t = 4\zeta(3) - 11/2 + [2 + 6\zeta(4) - (\zeta(3) - 7/8)^2]\varepsilon + O(\varepsilon^2)$, which is equivalent to the definition of a new, say $\widetilde{MS}$, scheme:

$$\mu^{2\varepsilon}_{\widetilde{MS}} = \mu^{2\varepsilon}_{\overline{MS}}[1 + (4\zeta(3) - 11/2)\varepsilon + [6\zeta(4) - 2(4\zeta(3) - 11/2)]\varepsilon^2 + O(\varepsilon^3)] \qquad (59)$$

then the N_f dependence and the $\zeta(3)$ terms cancel exactly at the 3-loop level. As a result $r_2 = 1/12$. Moreover, the additional term $6\zeta(4)$ in the definition (59) cancels $\zeta(4)$ terms in the hadronic vacuum polarization function at the 3-loop level. Within this scheme the four-loop correction is almost independent of the number of flavors. The full result for the R-ratio has the following simple form:

$$R^{\widetilde{MS}}(s) = 3\sum_f Q_f^2\left[1 + \frac{\alpha_s}{\pi} + \frac{1}{12}\left(\frac{\alpha_s}{\pi}\right)^2 - \left(\frac{\alpha_s}{\pi}\right)^3(16.2 \pm 0.5)\right] - \left(\sum_f Q_f\right)^2\left(\frac{\alpha_s}{\pi}\right)^3 1.2 + O(\alpha_s^4) \qquad (60)$$

where the remainder dependence on the number of flavors is within the small uncertainty ± 0.5 for all physically reasonable N_f and completely negligible for any phenomenological applications. The last term is also very small (within $0.4(\alpha_s/\pi)^3$) and negligible. The running coupling could be parametrised in the standard form (5) with $\Lambda_{\widetilde{MS}} = 1.41\Lambda_{\overline{MS}}$.

Using the "effective scheme" approach [55], let's rewrite (60) as:

$$R(s) = 3\sum_f Q_f^2\left[1 + \frac{\alpha_s^{eff}}{\pi} + O(\alpha_s^3)\right] \qquad (61)$$

where the 3-loop correction is absorbed into the definition of the effective coupling, which is given by (5) with Λ replaced by

$$\Lambda_{eff} = \Lambda_{\widetilde{MS}} exp\left(-\frac{1}{2\beta_0}\frac{r_2}{r_1}\right) \approx 1.02\Lambda_{\widetilde{MS}}$$

As we can see the new scheme $\widetilde{MS}$ almost coincides with the effective one and the fastest convergence is guaranteed, according to [55], within the wide range of energy defined by the renormalization group invariant criterion (note, Λ_{eff} is a renormalization group invariant quantity):

$$s/\Lambda^2_{eff} \sim s/\Lambda^2_{\widetilde{MS}} \gg 1$$

In the fig.8 we have plotted the dependence of the partial sums for R_τ on the t-parameter. We can see that according to "Minimal sensitivity" principle, t-parameter should be chosen within $t \in (-2.5, 0)$. If one chooses t according to $\widetilde{MS}$, then for $N_f = 3$ we get:

$$R_\tau^{\widetilde{MS}} = 3\left[1 + \frac{\alpha_s}{\pi} + 3.65\left(\frac{\alpha_s}{\pi}\right)^2 + 9.83\left(\frac{\alpha_s}{\pi}\right)^3\right] + O(\alpha_s^4) \tag{62}$$

Corrections are much smaller then in $\overline{MS}$ (see [31]). However, if we create a new scheme, connected with the τ-lepton: $t_\tau = t_{\widetilde{MS}} - 19/12 = 4\zeta(3) - 85/12$ we get:

$$R_\tau = 3\left[1 + \frac{\alpha_s}{\pi} + \frac{1}{12}\left(\frac{\alpha_s}{\pi}\right)^2 - \left(\frac{\alpha_s}{\pi}\right)^3 (6.1076 + 1.2704 N_f - 0.0141 N_f^2)\right] + O(\alpha_s^4) \tag{63}$$

The N_f dependence and the $\zeta(3)$-term cancel exactly at $O(\alpha_s^2)$.

6.4 Estimation of theoretical uncertainty

We will use our four-loop result for estimating the theoretical uncertainty of R. (For the similar analysis of perturbative calculations of coefficient functions in operator product expansions see: [26].) It should be stressed that within this new scheme the ratio r_3/r_2 is large and this could mean that the asymptotic series blows up at this level. In this situation it seems that one of the reasonable way is to assume that our series reaches its "optimal order" already at $O(\alpha_s^2)$. With this assumption we get for the QCD contribution in R:

$$\delta_{QCD}^{\widetilde{MS}} \equiv \frac{R^{\widetilde{MS}} - R_{QPM}}{R_{QPM}} = \frac{\alpha_s}{\pi} + \frac{1}{12}\left(\frac{\alpha_s}{\pi}\right)^2 + (\delta_{QCD}^{err} = 4\%) \tag{64}$$

There is an alternative way. We can include the calculated four-loop correction and estimate the theoretical error by the last included term. In this case we get:

$$\delta_{QCD}^{\widetilde{MS}} \equiv \frac{R^{\widetilde{MS}} - R_{QPM}}{R_{QPM}} = \frac{\alpha_s}{\pi} + \frac{1}{12}\left(\frac{\alpha_s}{\pi}\right)^2 + (16.2 \pm 0.5)\left(\frac{\alpha_s}{\pi}\right)^3 + (\delta_{QCD}^{err} = 4\%) \tag{65}$$

An analysis of fig.6 shows that the deviation of the four-loop approximant from the horizontal line is also about 4% within the reasonable range of t-parameter. This is consistent with Stevensons principle. Note also, that the estimation of 4% very weakly depends on the choice of scheme-scale within reasonable range of t-parameter.

It seems that the eq.(65) is most suitable for phenomenological applications. The comparison of the theoretical and experimental uncertainties shows that For the future, progress in QCD tests awaits more precise measurements.

7 Concluding Remarks

Using the current development of calculational methods, algorithms and computer programs for algebraic programming systems we have calculated the important quantity $R(s)$ in electron positron annihilation process at the four-loop level of perturbative QCD. Nowadays this is the most complicated analytical perturbative calculation of a physical quantity. The importance of this calculation has been motivated also because of the erroneous results of the previous calculations [25] with dramatic phenomenological consequences. Our result [27] is smaller by an order of magnitude and has the opposite sign compared to the wrong results of [25].

In the present paper we have described the basic methods of calculation and some of the key details. As an intermediate result of the calculation, we evaluate the four-loop QED β-function and, using the method first suggested by Braaten [29], we find the $O(\alpha_s^3)$ perturbative correction to the quantity $R_\tau = \Gamma(\tau^- \to \nu_\tau + hadrons)/\Gamma(\tau^- \to \nu_\tau e^- \overline{\nu}_e)$ in τ-decay.

For the $R(s)$ and R_τ quantities we have considered the scheme-scale ambiguity problem within 't Hooft's MS-prescription. We used the three most popular approaches for resolving the scheme-scale ambiguity. We find that Stevenson's minimal sensitivity principle works perfectly within a wide range of scheme-scale parameter. We fix the scale which simultaneously satisfies the Principle of minimal sensitivity, Brodsky-Lepage-Mackenzie approach and Grunberg's fastest apparent convergence principle. We have estimated the theoretical uncertainty of perturbative calculation of $R(s)$ at 4%.

Finally we should include the following notes. The recalculation of $R(s)$ and the QED β-function has also been done by the authors of the original calculation [25]. Their results completely agree with ours. However, as correctly concluded by A.Kataev in ref. [68], these two calculations are not completely independent in a following sense of this statement. Indeed, a few notes are in order. As we have mentioned above, the program [44], which has been used in the original calculation [25] turned out to be incorrect. It contained at least two sources of errors. One of them, which was due to an incorrect logical structure of the program [44] for trace calculations, has been

discovered by one of us (L.R.S.) while working with the program [16]. (Note, that the program [16] uses some ideas of [44] however, this program contains a number of original solutions and the structure has been changed according to the needs of IBM version of SCHOONSCHIP. Note also that the program [16] has been published in [15] with the name MINCER) Another source of errors has been discovered by A.Kataev, S.Larin and one of us (L.R.S.) with the help of test runs on the correct program [16]. The error was insufficient expansion in Laurent series of one-loop "basic" scalar integrals. (For more detailed discussion of the status of above programs see ref. [17].)

Thus the recalculation of $R(s)$ and QED β-function has been done by A.Kataev and S.Larin in [28, 52, 53] with the help of corrected program [44] [3] (Note that the wrong program block in [44] was substituted by the corresponding block of MINCER [16, 15]). Nevertheless the fact that these two calculations give identical results is strong evidence that the results are correct.

Acknowledgements We are grateful to N.S.Amaglobely, V.A.Matveev, V.A.Rubakov and A.N.Tavkhelidze for interest in this work and permanent support. One of us (L.R.S.) would like to express his gratitude to N.G.Deshpande and especially D.E.Soper for their support and valuable discussions. We would like to thank E.Braaten, S.Brodsky, S.Larin, J.Moffat, S.Nandi, P.Osland, A.Pivovarov, F.Tkachov,V.Tokarev, B.L.F.Ward and G.West for comments at various stages of this work. We also thank A.Kataev for his critical comments [69], where some of the typographical misprints in [27] are discussed.

It is pleasure to thank W.A.Bardeen and P.B.Mackenzie for their invitation and kind hospitality at the Theoretical Physics Department of Fermilab where the final version of this manuscript was completed.

This work was supported by the U.S. Department of Energy under grant No. DE-FG06-85ER-40224 and under grant No. DE-FG05-84ER40215.

[3]The authors of [28] claim that their calculation has been done by the program MINCER and refering on the paper [44]. However the paper [44] never mentiones the name "MINCER". The name MINCER first has apeared in the ref. [15]. The prehistory of above mentioned programs has been discussed in [17].

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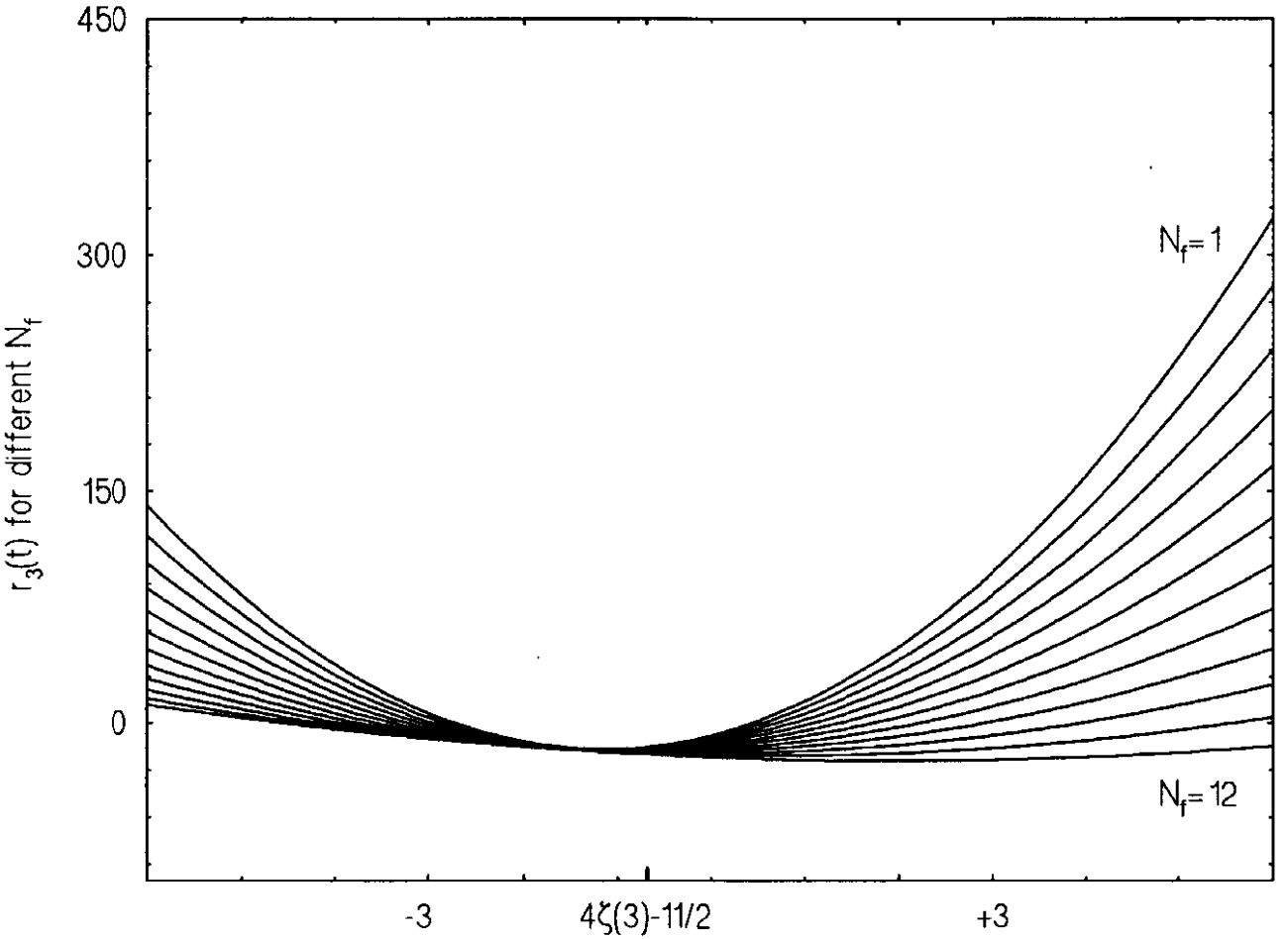

Fig. 6

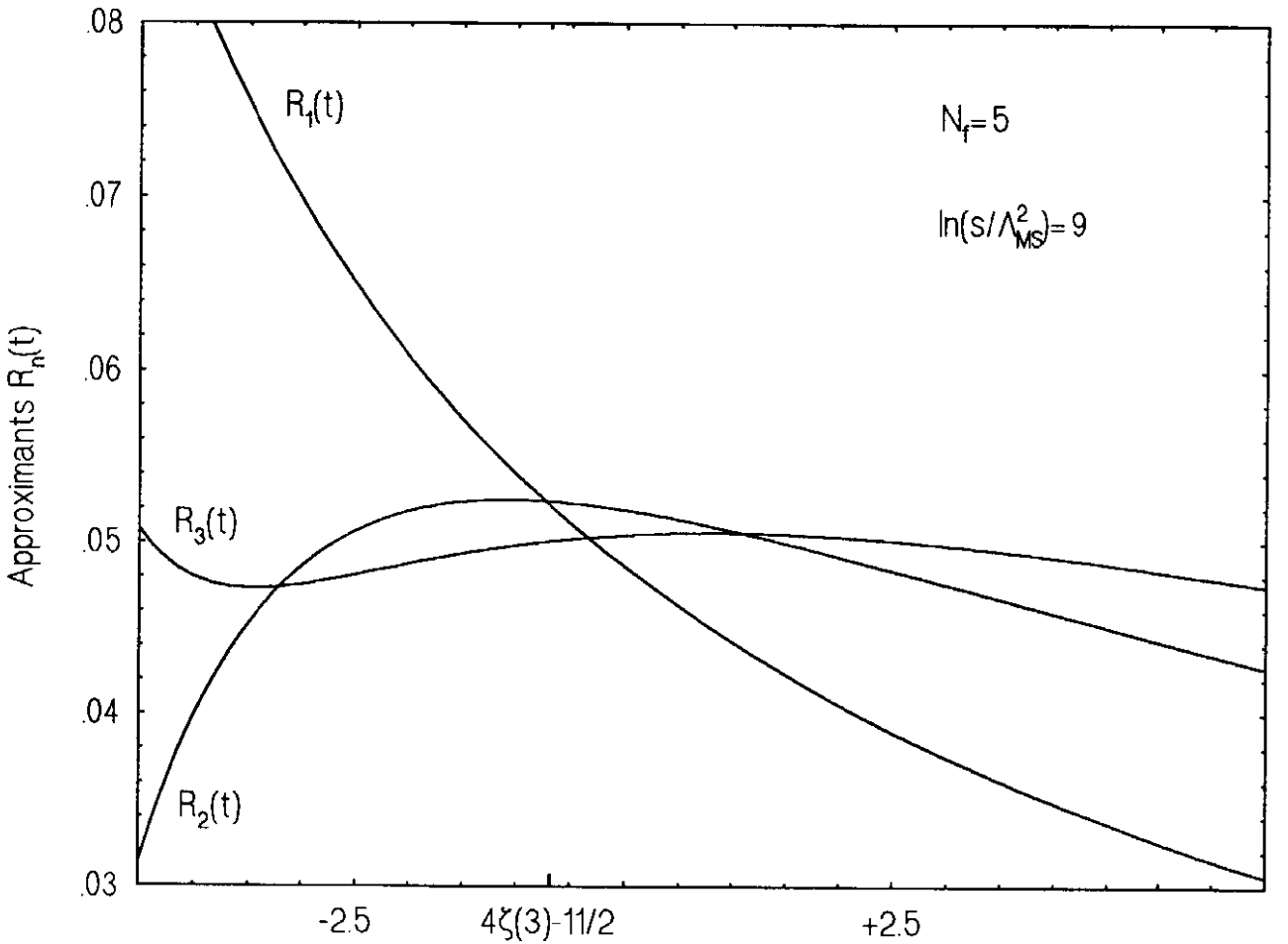
.08
.07
.06
.05
.04
.03
Approximants $R_n(t)$
$R_1(t)$
$R_3(t)$
$R_2(t)$
$N_f=5$
$\ln(s/\Lambda^2_{MS})=9$
-2.5
$4\zeta(3)-11/2$
+2.5

Fig.7

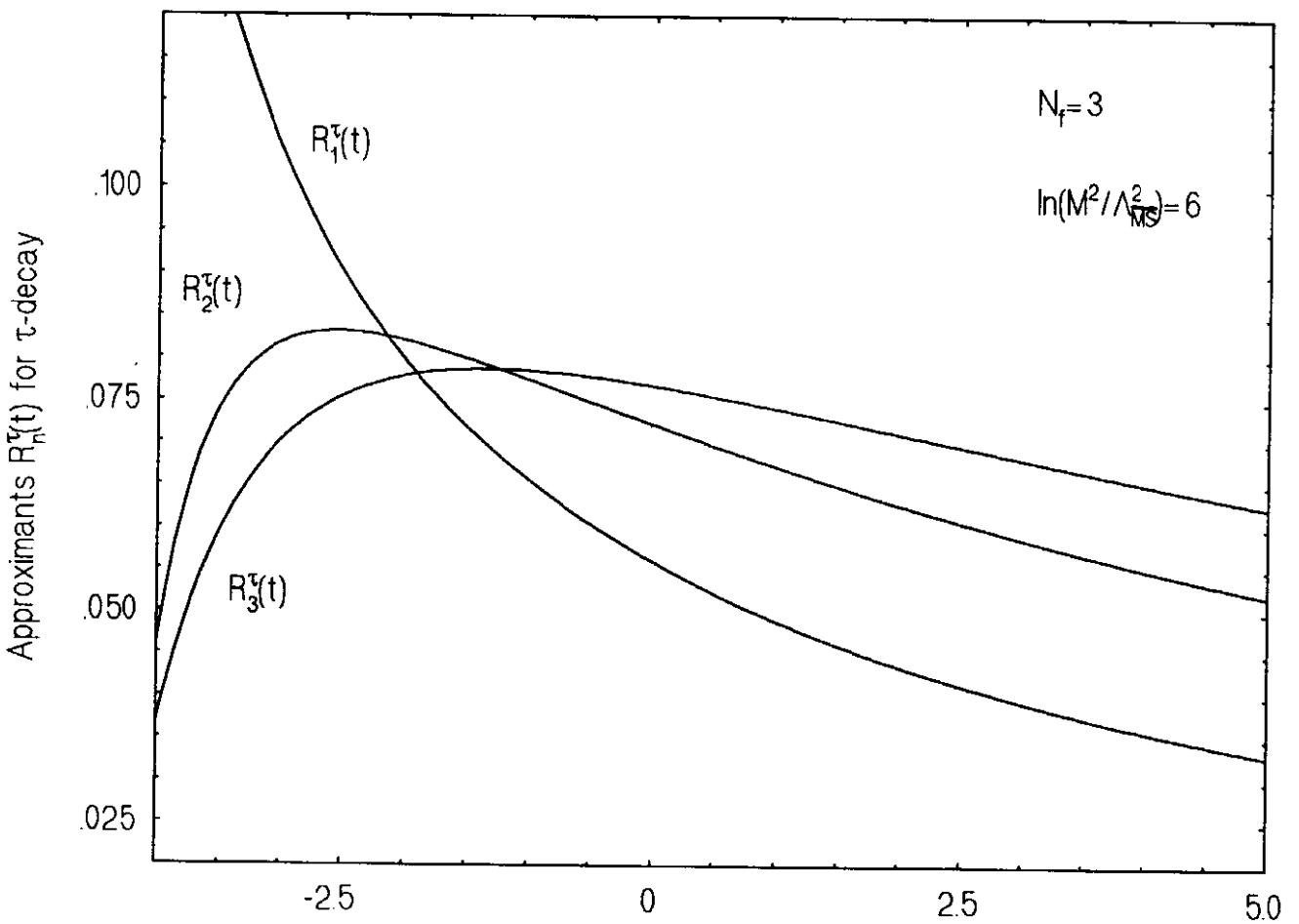
Approximants $R_n^\tau(t)$ for τ-decay
.100
.075
.050
.025
-2.5
0
2.5
5.0
$R_1^\tau(t)$
$R_2^\tau(t)$
$R_3^\tau(t)$
$N_f = 3$
$\ln(M^2/\Lambda^2_{\overline{MS}}) = 6$

Fig. s